Banach Dirac-Dual Dirac method for periodic cyclic homology

Axel Gastaldi, I2M

Abstract

The aim of this talk is to provides a natural identification between crossed product algebras with respect to a Lie group G and to a maximal compact subgroup $K < G$, extending globally a result of V. Nistor stated in the early 90's. We develop a Banach-analytic analogue of the Cuntz's approach to bivariant K-theory, and we adapt the Dirac-dual Dirac method for periodic cyclic homology. As a consequence, for all Banach G-algebra A, the following isomorphism occurs:

$$HP_\bullet(\mathcal{C}_c^\infty(G/K, A) \rtimes G) \simeq HP_{\bullet+\dim(G/K)}(A \rtimes G).$$

I) Setup

Let G be a Lie group over $\mathbb{R}$, and A a complex Banach algebra endowed with a smooth action of G. We define *the crossed product of A by G* as the following convolution algebra:

$$A \rtimes G := \mathcal{C}_c^\infty(G, A) \quad (f_1 \star f_2)(g) = \int_G f_1(h)h \cdot f_2(h^{-1}g)dh \in A.$$

Crossed products are powerful algebraic tools to understand covariant representations of the pair (G, A).

To study $A \rtimes G$, we will use periodic cyclic homology (HP), see [Lod] for more details. HP is a $\mathbb{Z}/2\mathbb{Z}$-graded theory such that :
→ If M is a compact manifold:

$$HP_0(\mathcal{C}^\infty(M)) = H_{DR}^{\text{even}}(M)$$
$$HP_1(\mathcal{C}^\infty(M)) = H_{DR}^{\text{odd}}(M)$$

→ It corresponds to the codomain of the Chern character:

$$K_i(A) \xrightarrow{\text{Ch.}} HP_i(A)$$

which leads to a pairing $\langle -, - \rangle : K \times HP \to \mathbb{C}$.
→ It is computed using a $\mathbb{Z}/2\mathbb{Z}$-graded complex:

$$\widehat{CC}(A) \Rightarrow HP(A) = H(\widehat{CC}(A))$$

Aim of the talk: Description of $HP(A \rtimes G)$.

II) Expectations

When $K < G$ is a maximal compact subgroup, the homogeneous space $G/K \sim \mathbb{R}^q$ is contractible so one may expect to obtain results relating G and the maximal compact subgroup K:

$$HP_\bullet(A \rtimes G) \overset{?}{\approx} HP_\bullet(A \rtimes K).$$

In our study, we will fix $q = 2m$ to be even. It is more convenient for degree shift modulo 2.

Theorem: (Nistor, 1993, [Nis]) For all Banach algebra A and $x \in G$:

$$HP_\bullet(A \rtimes G)_x \simeq HP_\bullet(A \rtimes K)_x.$$

Remark: The subscript means that we localize at a conjugacy class $\langle x\rangle$ of G (see [Bur] for the discrete case). More presicely, $C(A \rtimes G)_x$ corresponds to the subcomplex of smooth complactly supported functions lying on tuples of G whose product is closed to $\langle x\rangle$. It is the localization of the complex $C(A \rtimes G)$ at the prime ideal of functions wanishing at x of the algebra of central functions in G. This isomorphism is obtain as q compositions of coboundary maps but is functorial in A.

During the first part of my PhD I tried to make this isomorphism more natural, and of geometric nature to obtain, by gluing, a global statement. This strategy is hard to follow because in order to jump from a conjugacy class to another, we escape the cyclic world and the object are harder to manipulate. We try here another strategy: we decompose decompose the description using the intermediate algebra $\mathcal{C}_c^\infty(G/K, A) \rtimes G$ (comes from an idea of Xiang Tang).

III) Results

The first step is to related the followings

$$HP_\bullet(A \rtimes K) \overset{?}{\approx} HP_\bullet(\mathcal{C}_c^\infty(G/K, A) \rtimes G).$$

It may come from a Banach version of the Green's imprimitivity theorem (Green, 1976, [Gre]):

$$A \rtimes K \underset{\text{Morita}}{\sim} \mathcal{C}_0(G/K, A) \rtimes G,$$

but we still haven't the proof, yet.

The second step is to link:

$$HP_\bullet(\mathcal{C}_c^\infty(G/K, A) \rtimes G) \overset{?}{\approx} HP_\bullet(A \rtimes G).$$

The main tool: For any Banach algebra B, if we have:

1. a G-equivariant algebra homomorphism $\alpha : B \to A$;
2. a K-equivariant algebra homomorphism $\beta : A \to B$;
3. such that $\widehat{CC}(\alpha \rtimes K)$ and $\widehat{CC}(\beta \rtimes K)$ homotopic inverses,

then the algebra homomorphism $\alpha \rtimes G$ is an isomorphism in periodic cyclic homology:

$$\alpha \rtimes G : HP_\bullet(B \rtimes G) \xrightarrow{\sim} HP_\bullet(A \rtimes G).$$

Proof. The following diagram commutes:

$$\begin{array}{ccc} HP_\bullet(B \rtimes G)_x & \xrightarrow{\alpha \rtimes G} & HP_\bullet(A \rtimes G)_x \\ \simeq\downarrow & & \downarrow\simeq \\ HP_\bullet(B \rtimes K)_x & \underset{\beta \rtimes K}{\overset{\alpha \rtimes K}{\underset{\longleftarrow}{\overset{\longrightarrow}{\simeq}}}} & HP_\bullet(A \rtimes K)_x \end{array}$$

The top arrow becomes an isomorphism for all conjugacy class, and then a global isomorphism. QED.

In other words, it only requires a K-equivariant inverse of a global G-equivariant algebra homomorphism between B and A to get a global isomorphism in periodic cyclic homology.

Problem: The only algebra homomorphisms $\mathcal{C}_c^\infty(G/K, A) \to A$ we could get are only K-equivariant and not G-equivariant.
We then need to replace $\mathcal{C}_c^\infty(G/K, A)$ with another algebra which has an homotopy equivalent periodic cyclic complex and is equipped with a G-equivariant map with codomain A. We will be inspired by the Cuntz's identification but for $C^\star$-algebras.

Theorem: (Cuntz, 1986, [Cun]) For any $C^\star$-algebra $\mathcal{A}$ it exists a $C^\star$-algebra $q\mathcal{A} := \ker(\mathcal{A} \star \mathcal{A} \to \mathcal{A})$ such that:

- $KK_0(\mathcal{A}, \mathcal{B}) \simeq [q\mathcal{A}, \mathcal{B}\widehat{\otimes}_\pi \mathcal{K}]$ for all $C^\star$-algebra $\mathcal{B}$; we write $\alpha^\sharp$ for the class of a Kasparov bimodule α under this identification;
- Up to 2×2 matrices, $q^2\mathcal{B}$ and $q\mathcal{B}$ are homopy equivalent $\Rightarrow K_\bullet(q\mathcal{A}) \simeq K_\bullet(\mathcal{A})$;
- For $\alpha \in KK_0(\mathcal{A}, \mathcal{B})$ and $\beta \in KK_0(\mathcal{B}, \mathcal{C})$:

$$q\mathsf{A} \approx q(q\mathsf{A}) \xrightarrow{q\alpha^\sharp} q\mathsf{B} \xrightarrow{\beta^\sharp} \mathsf{C}, \qquad q(q\mathsf{A}) \xrightarrow{\langle\alpha,\beta\rangle^\sharp} \mathsf{C}$$

which is that $\langle\alpha, \beta\rangle^\sharp$ is homotopic to $\beta^\sharp \circ q\alpha^\sharp$.

This theorem means that the homotopy class of a composition of algebra homomorphisms may depends on the result of a Kasparov product.

Theorem: (G., 2025). If B is a Banach algebra there exists, for all $R \geqslant 1$, a Banach algebra $q_R B := \ker(B \star_R B \to B)$ (where $B \star_R B$ is obtained by completing $B \star B$ for some norms $\|\cdot\|_R$) such that:

- Any Banach Kasparov bimodule $\alpha := (H, \rho, F)$ over A induces, for $R \geqslant \|F\|^2$, an algebra homomorphism
$$\alpha^\sharp : q_R B \to \mathcal{K};$$
- Any idempotent matrix $\beta \in \mathcal{M}_m(B)$ induces, for $R \geqslant \|\beta\|$, an algebra homomorphism
$$\beta^\sharp : q_R\mathbb{C} \to B\widehat{\otimes}_\pi\mathcal{K};$$
- $\widehat{CC}(q_R B)$ and $\widehat{CC}(B)$ are homotopy equivalent for all $R \geqslant 1$;
- $\widehat{CC}(\alpha^\sharp) \circ \widehat{CC}(q_R\beta^\sharp) \approx \widehat{CC}(\langle\alpha, \beta\rangle^\sharp)$.

Compare to the Cuntz identification, we lost the possibility to compute the opposite-direction composition $\widehat{CC}(\beta^\sharp) \circ \widehat{CC}(q\alpha^\sharp)$ via Kasparov product. Indeed, in our circumstance of special G-varieties as G/K, we will use a Atiyah-Kasparov trick (see [Ati] and [Kas]) that enable to compute the Kasparov product $\langle\beta, \alpha\rangle$ whenever $\langle\alpha, \beta\rangle = 1$.

Corollary: If $B = \mathcal{C}_c^\infty(G/K) \simeq \mathcal{C}_c^\infty(\mathbb{R}^{2m})$ it exists a Banach **Dirac element** α and a **Bott element** β such that:

- $\alpha^\sharp$ is G-equivariant;
- $\beta^\sharp$ is K-equivariant;
- $\widehat{CC}((\alpha^\sharp \otimes id) \rtimes K)$ and $\widehat{CC}((q_R\beta^\sharp \otimes id) \rtimes K)$ homotopic inverse for R great enough.

Theorem: (G., 2025) The following algebra homomorphism is an isomorphism in periodic cyclic homology:

$$(\alpha^\sharp \otimes id) \rtimes G : HP_\bullet(\mathcal{C}_c^\infty(G/K, A) \rtimes G) \xrightarrow{\sim} HP_\bullet(A \rtimes G).$$

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